Shinji's Theorem

Ohad Lutzky & Nashimoto-Sempai Shinji

July 19, 2007

1 Introduction & Motivation

Again tormenting the reader with English to bring some logic insight, we present an alternative approach for dealing with what the author believes to be the single hardest subject taught in the Logic & Set Theory course for the CS faculty of the Technion: Definability. Proving a set to be definable is quite a bit of work in the 5-step method presented in class, and the usage of the compactness theorem can be quite confusing. Worse yet, the concept of a definable set is difficult to grasp, and there is very little intuition to lead us, given a set, to a conclusion as to whether it is definable or not.

I will present here a theorem which will link formal logic with calculus. Surprisingly enough, it can greatly simplify handling of definability. Our basic strategy will be defining a *limit* of assignments — this will not be difficult when viewing assignments as infinite binary vectors. We will then show that for a set of assignments to be definable, it has to, informally, contain all of its limits, and vice versa. This makes it easy to see that the set $K = \{z \in ASS | z \text{ gives only a finite number of atoms the value } 1\}$ is not definable, as, in a sense, it "converges" to $z_{\mathbf{T}}$, which assigns 1 to *all* atoms, and is therefore not in the set. The concept of a limit is relatively familiar, and will greatly assist viewing this simply.

While the theorem can be used to directly show definability (or, usually, lack thereof), it has a more important intuitive property: It is relatively intuitive to see whether or not a set of assignments is closed (contains all of its limits), and this can put a student on the right path to answering a prove/disprove question correctly, without making explicit use of the theorem.

2 Definitions

We will first begin with a definition of limits of sequences of assignments, similar in concept to the idea of a limit of sequences of real numbers.

Note 1. We will use the $\subseteq$ notation between sequences and sets to indicate that every member of the sequence is also a member of the set.

Definition 1. Let $\{z_n\}_{n=0}^{\infty} \subseteq ASS$ be a sequence of assignments in propositional calculus. We will say that the sequence is *convergent* and that it has the limit $\lim_{n\to\infty} z_n = z$ if for every $i \in \mathbb{N}$ there exists N_i such that if $n > N_i$, then z_n and z agree on the first i atoms, or formally, $\forall j \leq i, z_n(j) = z(j)$.

It's easy to see that one can define the limit equivalently by only checking that there exists N'_i for which $n > N'_i$ implies that $z_n(p_i) = z(p_i)$: if the previous definition holds, then $N'_i = N_i$ provides this condition. If this condition holds, $N_i = \max_{j \leq i} N'_j$ will give the previous one. This definition of a limit is more intuitive, but our first definition is used in our proofs.

Example 1. Consider the sequence $\{z_n\}_{n=0}^{\infty}$ in which z_n gives all of the atoms before p_n 1, and the rest 0. Formally,

$$z_n(p_i) = \begin{cases} 1, & i \leq n \\ 0, & i > n \end{cases}$$

If viewed as a sequence of infinite binary vectors, this sequence looks like this:

$$\begin{aligned} z_0 &= 1000000\ldots \\ z_1 &= 1100000\ldots \\ z_2 &= 1110000\ldots \\ &\vdots \end{aligned}$$

It is fairly easy to see that this sequence converges to $z_{\mathbf{T}}$, taking $N_i = i$. For each atom p_i, starting from the ith assignment, all of the assignments give 1 to p_i and all of the atoms before it, which is exactly what $z_{\mathbf{T}}$ would give. Formally, we take $N_i = i$. Immediately from our definition of the sequence, for all $n > i, j \leq i$, $z_n(p_j) = 1 = z_{\mathbf{T}}(p_j)$, and thus we have proven $\lim_{n\to\infty} = z_{\mathbf{T}}$ directly by definition.

An easier way to view this would be by the alternative definition of a limit. For each atom p_i, there is some point in the sequence at which all assignments start giving it 1 forever. Formally, we take $N'_i = i$, and again by the definition of our sequence, for all $n > N'_i$, $z_n(p_i) = 1 = z_{\mathbf{T}}(p_i)$. In a sense, for every i, $\lim_{n\to\infty} z_n(p_i) = 1 = z_{\mathbf{T}}(p_i)$, and this is an equivalent definition.

Definition 2. Let $K \subseteq ASS$ be a set of assignments. If every convergent sequence $\{z_n\}_{n=0}^{\infty} \subseteq K$ satisfies $\lim_{n\to\infty} z_n \in K$, we will say that K is *closed.*

Example 2. Take K_{fin}, that is, the set of assignments which give 1 to at most a finite number of atoms. We can see that, for the sequence $\{z_n\}_{n=0}^{\infty}$ we defined earlier, $\{z_n\}_{n=0}^{\infty} \subseteq K_{fin}$: Each assignment z_n gives only a finite number (n) of atoms the value 1. However, we've shown $\lim_{n\to\infty} z_n = z_{\mathbf{T}}$, and as $z_{\mathbf{T}} \notin K_{fin}$, we see that K_{fin} is not closed.

Theorem 1 (Shinji's Theorem)**.** *Any set of assignments K is closed if and only if it is definable.*

Example 3. K_{fin} is not closed, therefore it is not definable.

Here is an easy exercise (when using Shinji's theorem): Prove that for any $m \in \mathbb{N}$, the following set is not definable:

$$K_m = \{z \in ASS | z \text{ gives at least } m \text{ atoms the value } 1\}$$

3 Proof

We will prove Shinji's Theorem by proving the following two claims:

Claim 1. *Any definable set of assignments is closed.*

Claim 2. *Any closed set of assignments is definable.*

Proof of Claim 1. Let $\{z_n\}_{n=0}^{\infty} \subseteq K$ be a convergent sequence of assignments, and let $y = \lim_{n\to\infty} z_n$. Assume by contrast that $y \notin K$. K is definable by some set of formulae $\Phi \subseteq \mathbf{WFF}$. Therefore, since $y \notin K = MOD(\Phi)$, then $y \not\vDash \Phi$. Thus there exists $\varphi \in \Phi$ such that $y \not\vDash \varphi$. However, $\forall n, z_n \in K$, so $z_n \vDash \varphi$.

$\varphi \in \mathbf{WFF}$, and is therefore finite — thus, there exists a maximal index i of atoms which appear in φ. However, $\lim_{n\to\infty} z_n = y$, so there exists N_i such that for all $n > N_i$, z_n and y agree on the first i atoms. We know that $z_{N_i} \vDash \varphi$, and since z_{N_i} and y agree on all of the atoms which appear in φ, we have that $y \vDash \varphi$, a contradiction. □

Lemma 1. *If K is closed, then for any assignment $y \notin K$, there exists a formula φ_y^K such that $y \vDash \varphi_y^K$, but for all $z \in K$, $z \not\vDash \varphi_y^K$.*

Proof of Lemma 1. We will construct the sequence of formulae B_i to be the formulae which are satisfied exactly by assignments which agree with y on the first i atoms. Formally,

$$b_i = \begin{cases} p_i, & y(p_i) \\ \neg p_i, & \neg y(p_i) \end{cases}, \qquad B_i = \bigwedge_{j<i} b_j$$

Now we will show that there exists i for which $B_i = \varphi_y^K$. Since by construction, $\forall i \in \mathbb{N}, y \vDash B_i$, it remains to show that there exists such i that for all $z \in K$, $z \not\vDash B_i$.

Assume by contrast that such i does not exist. Therefore, for any $i \in \mathbb{N}$ there exists z_i such that $z_i \vDash B_i$. By definition of B_i, this means that z_i agrees with y on the first i atoms. Consider $\{z_i\}_{i=0}^{\infty}$ — we have shown that $\lim_{i\to\infty} z_i = y$. However, $y \notin K$, so this is a contradiction to K being closed. □

Proof of Claim 2. Assume K is closed. Let $Y = ASS \setminus K$. By Lemma 1, for all $y \in Y$ there exists φ_y^K such that $y \vDash \varphi_y^K$ and for all $z \in K$, $z \not\vDash \varphi_y^K$. Let $\Phi = \left\{\varphi_y^K \in \mathbf{WFF} | y \in Y\right\}$, and $\overline{\Phi} = \{\neg\varphi \in \mathbf{WFF} | \varphi \in \Phi\}$. $\overline{\Phi}$ defines K:

- If $z \in K$, then for all $\varphi \in \overline{\Phi}$, since $\varphi = \neg\varphi_y^K$ for some $y \in Y$, we know that $z \not\vDash \varphi_y^K$, thus $z \vDash \varphi$.
- If $z \vDash \overline{\Phi}$, then by construction of Φ, for all $y \in Y$, $z \not\vDash \varphi_y^K$. This means that for all $y \in Y$, $z \neq y$; therefore $z \notin Y$ which implies $z \in K$.

□